Revisiting Equal Sums

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Abstract: *This article revisits the Equal Sums problem and considers all possible solutions. Depending on the row/column total, various solutions are possible. Sometimes the value of E is unique, sometimes E is not, and in other cases a value of E is not possible. Looking at all possible row/column totals, surprisingly there is only one value that E cannot represent.*

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1 Introducing the Task

Consider the following problem: Without repetition, place the digits one through nine into the configuration of square tiles shown in Figure 1 so that $A+B+C = C+D+E = E+F+G = G+H+I$. Find the unique value of E.

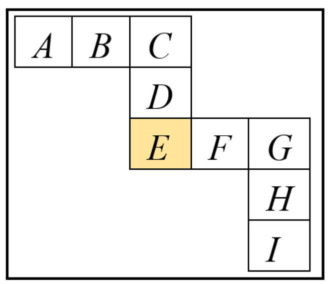

Fig. 1: *Row/column configuration.*

2 An Initial Exploration

After repeated attempts using systems of equations methods, it becomes evident that the row/column sum is vital. Letting the row/column total equal x and adding the rows and columns together yields (1).

$$(A+B+C)+(C+D+E)+(E+F+G)+(G+H+I) = 4x \tag{1}$$

Next, we rearrange the left side of this equation.

$$(A+B+C+D+E+F+G+H+I)+C+E+G = 4x \tag{2}$$

Since digits one through nine are represented within the parentheses, the sum equals 45.

$$45+C+E+G = 4x \tag{3}$$

Solving this equation for the variables (sum of C, E, and G) results in:

$$C + E + G = 4x - 45 \tag{4}$$

The smallest value for $4x - 45$ is $6 = 1 + 2 + 3$. Knowing that $4x - 45 \geq 6$, x must be larger than 12.75. Since the row/column total must be an integer, then $x \geq 13$. By letting the row/column total be thirteen and using the process of elimination, eight options are possible for the values of A through I (see Figure 2). A version of this case appears in *The Book of Mind Teasers & Mind Puzzlers* (Summers, 1986, p. 15). In each case, $E = 4$.

A	3	3	5	5	6	6	9	9
B	9	9	6	6	5	5	3	3
C	1	1	2	2	2	2	1	1
D	8	8	7	7	7	7	8	8
E	**4**	**4**	**4**	**4**	**4**	**4**	**4**	**4**
F	7	7	8	8	8	8	7	7
G	2	2	1	1	1	1	2	2
H	5	6	3	9	3	9	5	6
I	6	5	9	3	9	3	6	5
$A+B+C=$	13	13	13	13	13	13	13	13
$C+D+E=$	13	13	13	13	13	13	13	13
$E+F+G=$	13	13	13	13	13	13	13	13
$G+H+I=$	13	13	13	13	13	13	13	13

Fig. 2: *Row/Column totals of 13 and E value of 4.*

3 An Intriguing Follow-up Question

After arriving with a unique answer for E, the question arises, "What about other row/column totals?"

- If the row/column total is 14, there are five different values that E can represent (1, 2, 3, 7, & 8).
- Interestingly, if the row/column total is fifteen, there are no possible values for E. Surprisingly, $C+E+G$ will also equal fifteen. Therefore, $C+D+E = C+E+G$. Simplifying this equation results with $D = G$, but this is a contradiction since each letter must be a unique digit.
- There are again five values for E (2, 3, 7, 8, & 9) when the row/column total is sixteen.

A more generic version of this question appears in *The Inquisitive Problem Solver* (Vanderlind, Guy, & Larson, 2002, p. 20). The largest value that can be obtained by adding three unique digits is 24 (7 + 8 + 9). If the row/column total is 18, $C + E + G = 4(18)45 = 27$. Since this is not possible, the largest possible row/column total is seventeen, since $C + E + G$ must equal 23. By letting the row/column total equal 17 and using process of elimination, eight options are possible for the values of A through I (see Figure 3). In each case, $E = 6$.

A	1	1	4	4	5	5	7	7
B	7	7	5	5	4	4	1	1
C	9	9	8	8	8	8	9	9
D	2	2	3	3	3	3	2	2
$\mathbf{E}$	**6**	**6**	**6**	**6**	**6**	**6**	**6**	**6**
F	3	3	2	2	2	2	3	3
G	8	8	9	9	9	9	8	8
H	4	5	1	7	1	7	4	5
I	5	4	7	1	7	1	5	4
$A+B+C=$	17	17	17	17	17	17	17	17
$C+D+E=$	17	17	17	17	17	17	17	17
$E+F+G=$	17	17	17	17	17	17	17	17
$G+H+I=$	17	17	17	17	17	17	17	17

Fig. 3: *Row/column totals of 17 and E value of 6.*

4 Final Remarks

Looking back at the original problem, it can be noted that if given the row/column total is thirteen or sixteen it is possible to uniquely determine E. If given that the row/column total is 14, 15, or 16 it is not possible to uniquely determine E. (*Editors' note*: The author has offered to email possible solutions for the row/column total of 14 and 16 to interested readers.) After looking at the solutions obtained, a new interesting question arises. Given the original problem, what is the only value that E cannot be?

References

Summers, G. J. (1986). *The Great Book of Mind Teasers & Mind Puzzlers*. New York, NY: Sterling Publishing Company.

Vanderlind, P., Guy, R., & Larson, L. (2002). *The Inquisitive Problem Solver*. Washington, DC: The Mathematical Association of America.

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